

# A parameterized generalization of the sum formula for quadruple zeta values

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## Abstract

We give a parameterized generalization of the sum formula for quadruple zeta values. The generalization has four parameters, and is invariant under a cyclic group of order four. By substituting special values for the parameters, we also obtain weighted sum formulas for quadruple zeta values, which contain some known results.

## 1 Introduction

A multiple zeta value is a generalization of a classical special value of the Riemann zeta function  $\zeta(s) = \sum_{m=1}^{\infty} 1/m^s$ , and is defined by

$$\zeta(l_1, l_2, \dots, l_n) := \sum_{m_1 > m_2 > \dots > m_n > 0} \frac{1}{m_1^{l_1} m_2^{l_2} \cdots m_n^{l_n}} \quad (1.1)$$

for an index set  $(l_1, l_2, \dots, l_n)$  of positive integers with  $l_1 \geq 2$ . The integers  $l = l_1 + \dots + l_n$  and  $n$  are called the weight and the depth respectively. These values have arisen in various areas such as geometry, knot theory, number theory and mathematical physics [22]. There are many relations among these values, and an outstanding example is the sum formula which was proved for depth two by Euler [2], for depth three by Hoffman and Moen [9], and for general depth by Granville [4] and Zagier [23], independently. The formula says that the sum of all multiple zeta values of fixed weight  $l$  and depth  $n$  is expressed by the special value  $\zeta(l)$ , that is,

$$\sum_{\substack{l_1 \geq 2, l_2, \dots, l_n \geq 1 \\ (l_1 + \dots + l_n = l)}} \zeta(l_1, \dots, l_n) = \zeta(l). \quad (1.2)$$

Various generalizations of the sum formula have been studied: Ohno's relations, the cyclic, restricted and weighted sum formulas [1, 5, 8, 10, 14, 15, 16, 17, 18, 19, 20]. Recently parameterized generalizations of the sum formula, which we call parameterized sum formulas, were given for double and triple zeta values [3, 13]. The parameterized sum formula for double (resp. triple) zeta values has two (resp. three) parameters, and is invariant under a cyclic group of order two (resp. three), more precisely, the symmetry group  $S_2$  of degree two (resp. the alternating group  $A_3$  of degree three). By substituting special values for the parameters, these formulas yield some weighted sum formulas which contain the results of Ohno and Zudilin [18, Theorem 3] for double zeta values and of Guo and Xie [5, Theorem 1.1] for triple zeta values.

In this paper, we give a parameterized sum formula for quadruple zeta values which has four parameters and is invariant under a cyclic group of order four. By substituting special values for the parameters, we also obtain weighted sum formulas for quadruple zeta values which contain results of Guo and Xie [5] and of Ong, Eie and Liaw [19].

We prepare some notation in order to describe the parameterized sum formula precisely. Let  $S_n$  be the symmetric group of degree  $n$  and  $e$  its identity element, in particular, we put  $S = S_4$ . Let  $\langle \sigma_1, \dots, \sigma_m \rangle$  denote the subgroup generated by permutations  $\sigma_1, \dots, \sigma_m$ . Let  $C$  be the cyclic group  $\langle (1234) \rangle$  of order four, and  $\overline{C}$  the subset  $\{e, (1234)\}$  of  $C$ , where  $(i_1 \dots i_m)$  means a cyclic permutation defined by  $i_1 \mapsto \dots \mapsto i_m \mapsto i_1$ . Let  $H_\sigma$  stand for  $H\sigma$  for any subset  $H$  and element  $\sigma$  of  $S$ . We define a left action of  $S$  on the ring  $\mathbb{C}[x_1, x_2, x_3, x_4]$  of polynomials in four variables by  $\sigma \cdot f(x_1, x_2, x_3, x_4) := f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$  where  $\sigma \in S$  and  $f(x_1, x_2, x_3, x_4) \in \mathbb{C}[x_1, x_2, x_3, x_4]$ . We put  $x_{j_1 \dots j_m}^k = (x_{j_1} + \dots + x_{j_m})^k \in \mathbb{C}[x_1, x_2, x_3, x_4]$  for integers  $j_1, \dots, j_m, k$  with  $1 \leq j_a \leq 4, k \geq 0$ . For example, we have for  $\sigma, \rho, \nu \in S$ ,

$$x_{1234}^k = (x_1 + x_2 + x_3 + x_4)^k, \quad \sigma \rho \cdot x_{234}^k = \sigma \cdot x_{\rho(2)\rho(3)\rho(4)}^k = x_{\sigma\rho(2)\sigma\rho(3)\sigma\rho(4)}^k, \\ \sigma \cdot [x_{1234}^{k_1} x_{234}^{k_2} + x_{1\nu(3)}^{k_1} x_{\nu(3)2}^{k_2}] = x_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)}^{k_1} x_{\sigma(2)\sigma(3)\sigma(4)}^{k_2} + x_{\sigma(1)\sigma\nu(3)}^{k_1} x_{\sigma\nu(3)\sigma(2)}^{k_2}.$$

The parameterized sum formula for quadruple zeta values is as follows.

**THEOREM 1.1.** *Let  $l$  be an integer with  $l \geq 5$ , and  $x_1, x_2, x_3, x_4$  be parameters. Let  $\sum'$  mean running over all positive integers  $l_1, l_2, l_3, l_4$  satisfying  $l_1 \geq 2$  and  $l_1 + l_2 + l_3 + l_4 = l$ , and  $\zeta(\mathbf{l})$  mean  $\zeta(l_1, l_2, l_3, l_4)$ . Then we have*

$$\begin{aligned} & \sum' \left\{ \sum_{\sigma \in S} \sigma \cdot [x_{1234}^{l_1-1} x_{234}^{l_2-1} x_{34}^{l_3-1} x_4^{l_4-1}] \right. \\ & \quad - \sum_{\sigma \in C \cup \overline{C}(34)} \sigma \cdot \left[ \sum_{\rho \in \langle (234) \rangle} x_{134}^{l_1-1} x_{\rho(2)\rho(3)\rho(4)}^{l_2-1} x_{\rho(3)\rho(4)}^{l_3-1} x_{\rho(4)}^{l_4-1} \right. \\ & \quad \quad \left. + \sum_{\rho \in \langle (24) \rangle} x_{314}^{l_1-1} x_{14}^{l_2-1} x_{\rho(2)\rho(4)}^{l_3-1} x_{\rho(4)}^{l_4-1} + x_{341}^{l_1-1} x_{41}^{l_2-1} x_1^{l_3-1} x_2^{l_4-1} \right] \\ & \quad + \sum_{\sigma \in C} \sigma \cdot \left[ \sum_{\rho \in \langle (\nu(2)4) \rangle} \sum_{\nu \in \overline{C}} x_{1\nu(3)}^{l_1-1} x_{\nu(3)2}^{l_2-1} x_{\rho\nu(2)\rho(4)}^{l_3-1} x_{\rho(4)}^{l_4-1} + \sum_{\nu \in \overline{C}} x_{\nu(1)3}^{l_1-1} x_{2\nu(3)}^{l_2-1} x_{\nu(3)}^{l_3-1} x_{\nu(4)}^{l_4-1} \right. \\ & \quad \quad \left. + x_{41}^{l_1-1} x_1^{l_2-1} x_2^{l_3-1} x_3^{l_4-1} - x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1} \right] \left. \right\} \zeta(\mathbf{l}) \\ & = \left( \sum_{\substack{l_1, l_2, l_3, l_4 \geq 1 \\ (l_1 + l_2 + l_3 + l_4 = l)}} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1} \right) \zeta(l). \end{aligned} \quad (1.3)$$

We see that (1.3) is a straightforward generalization of the original sum formula because (1.3) with  $(x_1, x_2, x_3, x_4) = (1, 0, 0, 0)$  gives (1.2) for  $n = 4$ , and that (1.3) is invariant under  $C$  since  $\sum_{\sigma \in S}$  and  $\sum_{\sigma \in C_\alpha} (\alpha \in S)$  are unchanged under any left action of  $C$ .

As applications of the parameterized sum formula, we give the weighted sum formulas in Theorem 1.2 below, which consist of the known formulas (1.4), (1.5) and (1.6),

and the new formula (1.7) whose weights have not only powers of 2 but also powers of 3. To be exact, (1.4) and (1.6) were the results of [5, Theorem 1.1] and [19, Main Theorem] for quadruple zeta values, respectively. It seems that (1.5) is not printed out anywhere, but (1.5) is easily derived from subtracting (1.6) from twice (1.4).

**THEOREM 1.2** (cf. [5] and [19]). *Let  $l$ ,  $\sum'$  and  $\zeta(\mathbf{l})$  be as in Theorem 1.1.*

(i) *We have*

$$\sum' (2^{l_{123}-2} + 2^{l_{12}-2} + 2^{l_1-1} - 2^{l_{23}-1} - 2^{l_2-1}) \zeta(\mathbf{l}) = l \zeta(l), \quad (1.4)$$

$$\sum' (2^{l_{123}-1} + 2^{l_{12}-1} - 2^{l_{23}} - 2^{l_2} - 2^{l_3+1}) \zeta(\mathbf{l}) = (l-3) \zeta(l), \quad (1.5)$$

$$\sum' (2^{l_1} + 2^{l_3+1}) \zeta(\mathbf{l}) = (l+3) \zeta(l). \quad (1.6)$$

(ii) *We have*

$$\sum' (3^{l_2} 2^{l_1-1} - 3^{l_2} - 1) 2^{l_{13}} \zeta(\mathbf{l}) = \frac{(l+1)(l^2+5l-18)}{12} \zeta(l). \quad (1.7)$$

We outline the ways to prove the theorems. Theorem 1.1 is shown by a similar way adopted in [13], that is, we give some identities for multiple polylogarithms instead of multiple zeta values, and induce (1.3) from the identities by using asymptotic properties of multiple polylogarithms. Formulas (1.4) and (1.5) in Theorem 1.2 are directly proved by substituting special values for the parameters in (1.3), and (1.6) and (1.7) are induced from  $\mathbb{Q}$ -linear combinations of equations obtained by substitutions.

It is worth noting that we also obtain some weighted sum formulas whose weights are written in terms of powers of 2 and 3 in the course of the proof of (1.7), which do not have smart expressions like (1.7) (see Remark 4.2). Furthermore, we prove an equation about a cyclic sum of quadruple zeta values, which is related to Hoffman's result [7, Theorem 2.2], in giving the identities for multiple polylogarithms (see Remark 2.4).

The paper is organized as follows. In §2 which has three subsections, we discuss some facts about double, triple and quadruple polylogarithms as a preparation to prove the theorems. §3 and §4 devote the proofs of Theorems 1.1 and 1.2, respectively.

## 2 Parameterized sums of multiple polylogarithms

Let  $Li_{l_1, \dots, l_n}(z_1, \dots, z_n)$  be the multiple polylogarithm which is defined by

$$\begin{aligned} Li_{l_1, \dots, l_n}(z_1, \dots, z_n) &:= \sum_{m_1 > \dots > m_n > 0} \frac{z_1^{m_1-m_2} \cdots z_{n-1}^{m_{n-1}-m_n} z_n^{m_n}}{m_1^{l_1} \cdots m_{n-1}^{l_{n-1}} m_n^{l_n}} \\ &= \sum_{m_1, \dots, m_n > 0} \frac{z_1^{m_1} \cdots z_{n-1}^{m_{n-1}} z_n^{m_n}}{(m_1 + \cdots + m_n)^{l_1} \cdots (m_{n-1} + m_n)^{l_{n-1}} m_n^{l_n}} \end{aligned} \quad (2.1)$$

for an index set  $(l_1, \dots, l_n)$  of positive integers and a  $n$ -tuple  $(z_1, \dots, z_n)$  of complex numbers with  $|z_j| < 1$ . We define parameterized sums of double, triple and quadruple polylogarithms by

$$\mathfrak{DL}_l(x_1, x_2; z_1, z_2) := \sum^{\dagger} x_1^{l_1-1} x_2^{l_2-1} Li_l(z_1, z_2), \quad (2.2)$$

$$\begin{aligned}\mathfrak{TL}_l(x_1, x_2, x_3; z_1, z_2, z_3) &:= \sum^{\dagger} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} Li_l(z_1, z_2, z_3), \\ \mathfrak{QL}_l(x_1, x_2, x_3, x_4; z_1, z_2, z_3, z_4) &:= \sum^{\dagger} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1} Li_l(z_1, z_2, z_3, z_4),\end{aligned}$$

where  $\sum^{\dagger}$  means running over all positive integers  $l_1, \dots, l_n$  with  $l = l_1 + \dots + l_n$  and  $1$  does  $(l_1, \dots, l_n)$  for suitable  $n$ . We also denote  $x^{l-1} Li_l(z)$  by  $\mathfrak{SL}_l(x; z)$  for convenience. We note that  $\sum^{\dagger}$  and  $\sum'$  are different;  $\sum^{\dagger}$  contains  $l_1 = 1$  but  $\sum'$  does not.

In this section, we give some identities for the above parameterized sums, and calculate constant terms of asymptotic expansions of functions appearing in the identities. This section has three subsections; The first and second subsections devote the proofs of the identities, which are derived from harmonic and shuffle relations (see the subsections for details of the relations). In the third subsection, we calculate constant terms.

## 2.1 An identity derived from harmonic relations

The purpose of this subsection is to prove the following identity. We will give the proof in the end of this subsection.

**PROPOSITION 2.1.** *Let  $\langle z \rangle^n$  denote the  $n$ -tuple  $(z, z^2, \dots, z^n)$ , and  $l$  be an integer with  $l \geq 4$ . Then we have*

$$\begin{aligned}& - \sum_{\sigma \in C} \sigma \cdot \mathfrak{QL}_l(x_1, x_2, x_3, x_4; \langle z \rangle^4) \\& + \sum_{\substack{a \geq 3, b \geq 1 \\ (a+b=l)}} \sum_{\sigma \in C} \sigma \cdot [\mathfrak{TL}_a(x_1, x_2, x_3; \langle z \rangle^3) \mathfrak{SL}_b(x_4; z)] \\& + \sum_{\substack{a, b \geq 2 \\ (a+b=l)}} \sum_{\sigma \in \overline{C}} \sigma \cdot [\mathfrak{DL}_a(x_1, x_2; \langle z \rangle^2) \mathfrak{DL}_b(x_3, x_4; \langle z \rangle^2)] \\& - \sum_{\substack{a \geq 2, b, c \geq 1 \\ (a+b+c=l)}} \sum_{\sigma \in C} \sigma \cdot [\mathfrak{DL}_a(x_1, x_2; \langle z \rangle^2) \mathfrak{SL}_b(x_3; z) \mathfrak{SL}_c(x_4; z)] \\& + \sum_{\substack{a, b, c, d \geq 1 \\ (a+b+c+d=l)}} \mathfrak{SL}_a(x_1; z) \mathfrak{SL}_b(x_2; z) \mathfrak{SL}_c(x_3; z) \mathfrak{SL}_d(x_4; z) \\& = \left( \sum_{\substack{l_1, l_2, l_3, l_4 \geq 1 \\ (l_1+l_2+l_3+l_4=l)}} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1} \right) Li_l(z^4).\end{aligned}\tag{2.3}$$

We prepare some notation for the discussion after this. Let  $A (\subset S)$  be the alternating group of degree four. Let  $\sigma \cdot (f_1, \dots, f_n)$  denote  $(\sigma \cdot f_1, \dots, \sigma \cdot f_n)$  for any permutation  $\sigma$  of  $S$  and ordered set  $(f_1, \dots, f_n)$  of polynomials in four variables. We especially consider the case that each  $f_j$  is expressed by  $l_{j_1 \dots j_m}$  ( $1 \leq j_a \leq 4$ ), where  $l_1, l_2, l_3, l_4$  are positive integers. For a multiple polylogarithm  $Li_l(z, w, \dots)$  having such an ordered set  $\mathbf{l}$ , we set  $\sigma \cdot Li_l(z, w, \dots) := Li_{\sigma \cdot \mathbf{l}}(z, w, \dots)$ , and extended it to the  $\mathbb{Q}$ -algebra spanned by  $Li_l(z, w, \dots)$ 's naturally. For example,

$$\sigma \cdot [Li_{l_{12}, l_{34}}(z^2, z^4) + Li_{l_{123}}(z^3) Li_{l_4}(z)] = Li_{\sigma \cdot (l_{12}, l_{34})}(z^2, z^4) + Li_{\sigma \cdot l_{123}}(z^3) Li_{\sigma \cdot l_4}(z)$$

$$= Li_{l_{\sigma(1)\sigma(2)}, l_{\sigma(3)\sigma(4)}}(z^2, z^4) + Li_{l_{\sigma(1)\sigma(2)\sigma(3)}}(z^3) Li_{l_{\sigma(4)}}(z).$$

The key relations for the proof of (2.3) are the following harmonic relations which are derived from decomposition of summation.

**LEMMA 2.2.** *Let  $l_1, l_2, l_3, l_4$  be positive integers.*

(i) *We have*

$$\begin{aligned} Li_{l_1, l_2, l_3}(\langle z \rangle^3) Li_{l_4}(z) &= \sum_{\sigma \in \mathcal{U}_1} \sigma \cdot Li_{l_1, l_2, l_3, l_4}(\langle z \rangle^4) \\ &+ (243) \cdot [Li_{l_{12}, l_3, l_4}(z^2, z^3, z^4) + Li_{l_1, l_{23}, l_4}(z, z^3, z^4)] + Li_{l_1, l_2, l_{34}}(z, z^2, z^4), \end{aligned} \quad (2.4)$$

where

$$\mathcal{U}_1 = \{e, (34), (243), (1432)\}. \quad (2.5)$$

(ii) *We have*

$$\begin{aligned} Li_{l_1, l_2}(\langle z \rangle^2) Li_{l_3, l_4}(\langle z \rangle^2) &= \sum_{\sigma \in \mathcal{U}_2} \sigma \cdot Li_{l_1, l_2, l_3, l_4}(\langle z \rangle^4) \\ &+ \sum_{\sigma \in \mathcal{V}_2} \sigma \cdot [Li_{l_{12}, l_3, l_4}(z^2, z^3, z^4) + Li_{l_1, l_{23}, l_4}(z, z^3, z^4) + Li_{l_1, l_2, l_{34}}(z, z^2, z^4)] \\ &+ (23) \cdot Li_{l_{12}, l_{34}}(z^2, z^4), \end{aligned} \quad (2.6)$$

where

$$\mathcal{V}_2 = \{(23), (1342)\}, \quad \mathcal{U}_2 = \mathcal{V}_2 \cup \{e, (13)(24), (132), (234)\}. \quad (2.7)$$

(iii) *We have*

$$\begin{aligned} Li_{l_1, l_2}(\langle z \rangle^2) Li_{l_3}(z) Li_{l_4}(z) &= \sum_{\sigma \in \mathcal{U}_3} \sigma \cdot Li_{l_1, l_2, l_3, l_4}(\langle z \rangle^4) \\ &+ \sum_{\sigma \in \mathcal{V}_3^a} \sigma \cdot Li_{l_{12}, l_3, l_4}(z^2, z^3, z^4) + \sum_{\sigma \in \mathcal{V}_3^b} \sigma \cdot Li_{l_1, l_{23}, l_4}(z, z^3, z^4) + \sum_{\sigma \in \mathcal{V}_3^c} \sigma \cdot Li_{l_1, l_2, l_{34}}(z, z^2, z^4) \\ &+ \sum_{\sigma \in \mathcal{W}_3} \sigma \cdot Li_{l_{12}, l_{34}}(z^2, z^4) + (24) \cdot Li_{l_{123}, l_4}(z^3, z^4) + Li_{l_1, l_{234}}(z, z^4), \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} \mathcal{W}_3 &= \{(23), (24)\}, \quad \mathcal{V}_3 = \mathcal{W}_3 \cup \{(34), (1342), (1423), (1432)\}, \\ \mathcal{V}_3^a &= \mathcal{V}_3 \setminus \{(34)\}, \quad \mathcal{V}_3^b = \mathcal{V}_3 \setminus \{(1432)\}, \quad \mathcal{V}_3^c = \mathcal{V}_3 \setminus \{(1423)\}, \\ \mathcal{U}_3 &= \mathcal{V}_3 \cup \{e, (13)(24), (132), (142), (234), (243)\}. \end{aligned} \quad (2.9)$$

(iv) *We have*

$$\begin{aligned} Li_{l_1}(z) Li_{l_2}(z) Li_{l_3}(z) Li_{l_4}(z) &= \sum_{\sigma \in S} \sigma \cdot Li_{l_1, l_2, l_3, l_4}(\langle z \rangle^4) \\ &+ \sum_{\sigma \in A} \sigma \cdot [Li_{l_{12}, l_3, l_4}(z^2, z^3, z^4) + Li_{l_1, l_{23}, l_4}(z, z^3, z^4) + Li_{l_1, l_2, l_{34}}(z, z^2, z^4)] \\ &+ \sum_{\sigma \in C \cup \overline{C}_{(14)}} \sigma \cdot Li_{l_{12}, l_{34}}(z^2, z^4) + \sum_{\sigma \in C} \sigma \cdot [Li_{l_{123}, l_4}(z^3, z^4) + Li_{l_1, l_{234}}(z, z^4)] + Li_{l_{1234}}(z^4). \end{aligned} \quad (2.10)$$

*Proof.* By (2.1), we have

$$Li_{l_1, l_2, l_3}(z_1, z_2, z_3) Li_{l_4}(z_4) = \sum_{\substack{m_1 > m_2 > m_3 > 0 \\ m_4 > 0}} \frac{z_1^{m_1 - m_2} z_2^{m_2 - m_3} z_3^{m_3} z_4^{m_4}}{m_1^{l_1} m_2^{l_2} m_3^{l_3} m_4^{l_4}}.$$

From this and the decomposition of the summation

$$\begin{aligned} \sum_{\substack{m_1 > m_2 > m_3 > 0 \\ m_4 > 0}} &= \sum_{m_1 > m_2 > m_3 > m_4 > 0} + \sum_{m_1 > m_2 > m_4 > m_3 > 0} + \sum_{m_1 > m_4 > m_2 > m_3 > 0} + \sum_{m_4 > m_1 > m_2 > m_3 > 0} \\ &+ \sum_{m_1 = m_4 > m_2 > m_3 > 0} + \sum_{m_1 > m_2 = m_4 > m_3 > 0} + \sum_{m_1 > m_2 > m_3 = m_4 > 0}, \end{aligned}$$

we see that

$$\begin{aligned} Li_{l_1, l_2, l_3}(z_1, z_2, z_3) Li_{l_4}(z_4) &= Li_{l_1, l_2, l_3, l_4}(z_1, z_2, z_3, z_{34}^*) + Li_{l_1, l_2, l_4, l_3}(z_1, z_2, z_{24}^*, z_{34}^*) \\ &+ Li_{l_1, l_4, l_2, l_3}(z_1, z_{14}^*, z_{24}^*, z_{34}^*) + Li_{l_4, l_1, l_2, l_3}(z_4, z_{14}^*, z_{24}^*, z_{34}^*) \\ &+ Li_{l_{14}, l_2, l_3}(z_{14}^*, z_{24}^*, z_{34}^*) + Li_{l_1, l_{24}, l_3}(z_1, z_{24}^*, z_{34}^*) + Li_{l_1, l_2, l_{34}}(z_1, z_2, z_{34}^*), \end{aligned} \quad (2.11)$$

where we put  $z_{j_1 \dots j_m}^* = z_{j_1} \cdots z_{j_m}$ . By (2.11) with  $(z_1, z_2, z_3, z_4) = (z, z^2, z^3, z)$  and Table 1, we obtain (2.4).

As the same way, we see from the decomposition of  $\sum_{\substack{m_1 > m_2 > 0 \\ m_3 > m_4 > 0}}$  that

$$\begin{aligned} Li_{l_1, l_2}(z_1, z_2) Li_{l_3, l_4}(z_3, z_4) &= Li_{l_1, l_2, l_3, l_4}(z_1, z_2, z_{23}^*, z_{24}^*) + Li_{l_1, l_3, l_2, l_4}(z_1, z_{13}^*, z_{23}^*, z_{24}^*) \\ &+ Li_{l_1, l_3, l_4, l_2}(z_1, z_{13}^*, z_{14}^*, z_{24}^*) + Li_{l_3, l_1, l_2, l_4}(z_3, z_{13}^*, z_{23}^*, z_{24}^*) + Li_{l_3, l_1, l_4, l_2}(z_3, z_{13}^*, z_{14}^*, z_{24}^*) \\ &+ Li_{l_3, l_4, l_1, l_2}(z_3, z_4, z_{14}^*, z_{24}^*) + Li_{l_{13}, l_2, l_4}(z_{13}^*, z_{23}^*, z_{24}^*) + Li_{l_{13}, l_4, l_2}(z_{13}^*, z_{14}^*, z_{24}^*) \\ &+ Li_{l_1, l_{23}, l_4}(z_1, z_{23}^*, z_{24}^*) + Li_{l_3, l_{14}, l_2}(z_3, z_{14}^*, z_{24}^*) + Li_{l_1, l_3, l_{24}}(z_1, z_{13}^*, z_{24}^*) \\ &+ Li_{l_3, l_1, l_{24}}(z_3, z_{13}^*, z_{24}^*) + Li_{l_{13}, l_{24}}(z_{13}^*, z_{24}^*), \end{aligned} \quad (2.12)$$

which with  $(z_1, z_2, z_3, z_4) = (z, z^2, z, z^2)$  and Table 1 proves (2.6).

We will verify (2.8) by using (2.6). For this we need the following harmonic relations which are obtained by the decomposition of  $\sum_{m_1 > 0}$  and the one of  $\sum_{\substack{m_1 > m_2 > 0 \\ m_3 > 0}}$ ;

$$Li_{k_1}(w_1) Li_{k_2}(w_2) = Li_{k_1, k_2}(w_1, w_{12}^*) + Li_{k_2, k_1}(w_2, w_{12}^*) + Li_{k_{12}}(w_{12}^*), \quad (2.13)$$

$$\begin{aligned} Li_{k_1, k_2}(w_1, w_2) Li_{k_3}(w_3) &= Li_{k_1, k_2, k_3}(w_1, w_2, w_{23}^*) \\ &+ Li_{k_1, k_3, k_2}(w_1, w_{13}^*, w_{23}^*) + Li_{k_3, k_1, k_2}(w_3, w_{13}^*, w_{23}^*) \\ &+ Li_{k_{13}, k_2}(w_{13}^*, w_{23}^*) + Li_{k_1, k_{23}}(w_1, w_{23}^*). \end{aligned} \quad (2.14)$$

By (2.13) with  $(w_1, w_2) = (z, z)$  and  $(k_1, k_2) = (l_3, l_4)$ , and (2.14) with  $(w_1, w_2, w_3) = (z, z^2, z^2)$  and  $(k_1, k_2, k_3) = (l_1, l_2, l_{34})$ , we obtain

$$Li_{l_3}(z) Li_{l_4}(z) = \sum_{\sigma \in \langle (34) \rangle} \sigma \cdot Li_{l_3, l_4}(\langle z \rangle^2) + Li_{l_{34}}(z^2), \quad (2.15)$$

$$\begin{aligned} Li_{l_1, l_2}(\langle z \rangle^2) Li_{l_{34}}(z^2) &= (1423) \cdot Li_{l_{12}, l_3, l_4}(z^2, z^3, z^4) \\ &+ (24) \cdot Li_{l_1, l_{23}, l_4}(z, z^3, z^4) + (34) \cdot Li_{l_1, l_2, l_{34}}(z, z^2, z^4) \end{aligned} \quad (2.16)$$

Table 1: Each row of the lists gives the conditions of  $1$  and  $\sigma$  for  $1 = \sigma \cdot l'$ , where  $l' \in \{(l_{12}, l_3, l_4), (l_1, l_{23}, l_4), (l_1, l_2, l_{34}), (l_{12}, l_{34}), (l_{123}, l_4), (l_1, l_{234})\}$ .

$l' = (l_{12}, l_3, l_4)$		$l' = (l_1, l_{23}, l_4)$		$l' = (l_1, l_2, l_{34})$	
1	$\sigma$	1	$\sigma$	1	$\sigma$
$(l_{12}, l_3, l_4)$	$e, (12)$	$(l_1, l_{23}, l_4)$	$e, (23)$	$(l_1, l_2, l_{34})$	$e, (34)$
$(l_{12}, l_4, l_3)$	$(12)(34), (34)$	$(l_1, l_{24}, l_3)$	$(243), (34)$	$(l_1, l_3, l_{24})$	$(234), (23)$
$(l_{13}, l_2, l_4)$	$(132), (23)$	$(l_1, l_{34}, l_2)$	$(234), (24)$	$(l_1, l_4, l_{23})$	$(243), (24)$
$(l_{13}, l_4, l_2)$	$(234), (1342)$	$(l_2, l_{13}, l_4)$	$(123), (12)$	$(l_2, l_1, l_{34})$	$(12)(34), (12)$
$(l_{14}, l_2, l_3)$	$(243), (1432)$	$(l_2, l_{14}, l_3)$	$(12)(34), (1243)$	$(l_2, l_3, l_{14})$	$(123), (1234)$
$(l_{14}, l_3, l_2)$	$(142), (24)$	$(l_2, l_{34}, l_1)$	$(124), (1234)$	$(l_2, l_4, l_{13})$	$(124), (1243)$
$(l_{23}, l_1, l_4)$	$(123), (13)$	$(l_3, l_{12}, l_4)$	$(132), (13)$	$(l_3, l_1, l_{24})$	$(132), (1342)$
$(l_{23}, l_4, l_1)$	$(134), (1234)$	$(l_3, l_{14}, l_2)$	$(13)(24), (1342)$	$(l_3, l_2, l_{14})$	$(134), (13)$
$(l_{24}, l_1, l_3)$	$(143), (1243)$	$(l_3, l_{24}, l_1)$	$(134), (1324)$	$(l_3, l_4, l_{12})$	$(13)(24), (1324)$
$(l_{24}, l_3, l_1)$	$(124), (14)$	$(l_4, l_{12}, l_3)$	$(143), (1432)$	$(l_4, l_1, l_{23})$	$(142), (1432)$
$(l_{34}, l_1, l_2)$	$(13)(24), (1423)$	$(l_4, l_{13}, l_2)$	$(142), (1423)$	$(l_4, l_2, l_{13})$	$(143), (14)$
$(l_{34}, l_2, l_1)$	$(14)(23), (1324)$	$(l_4, l_{23}, l_1)$	$(14)(23), (14)$	$(l_4, l_3, l_{12})$	$(14)(23), (1423)$

$l' = (l_{12}, l_{34})$	
1	$\sigma$
$(l_{12}, l_{34})$	$e, (12)(34), (12), (34)$
$(l_{13}, l_{24})$	$(132), (234), (23), (1342)$
$(l_{14}, l_{23})$	$(142), (243), (24), (1432)$
$(l_{23}, l_{14})$	$(123), (134), (13), (1234)$
$(l_{24}, l_{13})$	$(124), (143), (14), (1243)$
$(l_{34}, l_{12})$	$(13)(24), (14)(23), (1423), (1324)$

$l' = (l_{123}, l_4)$		$l' = (l_1, l_{234})$	
1	$\sigma$	1	$\sigma$
$(l_{123}, l_4)$	$e, (123), (132), (12), (13), (23)$	$(l_1, l_{234})$	$e, (234), (243), (23), (24), (34)$
$(l_{124}, l_3)$	$(12)(34), (143), (243), (34), (1243), (1432)$	$(l_2, l_{134})$	$(12)(34), (123), (124), (12), (1234), (1243)$
$(l_{134}, l_2)$	$(13)(24), (142), (234), (24), (1342), (1423)$	$(l_3, l_{124})$	$(13)(24), (132), (134), (13), (1324), (1342)$
$(l_{234}, l_1)$	$(14)(23), (124), (134), (14), (1234), (1324)$	$(l_4, l_{123})$	$(14)(23), (142), (143), (14), (1423), (1432)$

$$+ (24) \cdot Li_{l_{123}, l_4}(z^3, z^4) + Li_{l_1, l_{234}}(z, z^4).$$

By (2.15), we also get

$$\begin{aligned} & Li_{l_1, l_2}(\langle z \rangle^2) Li_{l_3}(z) Li_{l_4}(z) \\ &= \sum_{\sigma \in \langle (34) \rangle} \sigma \cdot [Li_{l_1, l_2}(\langle z \rangle^2) Li_{l_3, l_4}(\langle z \rangle^2)] + Li_{l_1, l_2}(\langle z \rangle^2) Li_{l_{34}}(z^2). \end{aligned} \quad (2.17)$$

Since direct calculations show that

$$\begin{aligned} \langle (34) \rangle \cdot \mathcal{U}_2 &= \mathcal{U}_3, \\ \langle (34) \rangle \cdot \mathcal{V}_2 &= \{(23), (1342), (243), (142)\}, \\ \langle (34) \rangle \cdot (23) &= \{(23), (243)\}, \end{aligned}$$

(2.6), (2.16) and (2.17) with Table 1 prove (2.8).

We verify (2.10) finally. To exactly express the decomposition of a summation by permutations is difficult in general, but the case of  $\sum_{m_1, m_2, m_3, m_4 > 0}$  is not since the summation is invariant under  $S$  and its decomposition can be understood in terms of quotient sets of  $S$  as we see below. For an odd permutation  $(ij)$ ,  $A$  is a transversal of  $S/\langle (ij) \rangle$  since the numbers of  $A$  and  $S/\langle (ij) \rangle$  are equal and the canonical projection of  $A$  into  $S/\langle (ij) \rangle$  is injective. It also follows from Table 1 that  $C \cup \overline{C}_{(14)}$  is a transversal

of  $S/\langle(12), (34)\rangle$ , since  $C = \{e, (1234), (13)(24), (1432)\}$ ,  $\overline{C}_{(14)} = \{e, (1234)\}(14) = \{(14), (234)\}$ , and  $S/\langle(12), (34)\rangle$  is isomorphic to  $\{(l_{j_1j_2}, l_{j_3j_4}) \mid 1 \leq j_a \leq 4, j_a \neq j_b (a \neq b)\}/\sim$  by representation theory, where we say that  $\mathbf{l}_1 \sim \mathbf{l}_2$  for  $\mathbf{l}_i \in \{(l_{j_1j_2}, l_{j_3j_4})\}$  if and only if there is a permutation  $\sigma$  such that  $\mathbf{l}_1 = \sigma \cdot \mathbf{l}_2$ . Similarly it holds that  $C$  is a transversal of  $S/S^i$ , where  $S^i$  denotes  $\{\sigma \in S \mid \sigma(i) = i\}$  and  $i = 1, 4$ . Let  $\mathbb{N}$  be the set of positive integers and  $\mathbf{m}$  mean a lattice point  $(m_1, m_2, m_3, m_4)$  in  $\mathbb{N}^4$ . Since  $A, C \cup \overline{C}_{(14)}$  and  $C$  are transversals of certain quotient sets, the lattice points in  $\mathbb{N}^4$  decompose into the following disjoint subsets;

$$\begin{aligned} \{\mathbf{m} \mid m_{\sigma(1)} > m_{\sigma(2)} > m_{\sigma(3)} > m_{\sigma(4)}\}, & \quad \{\mathbf{m} \mid m_{\tau(1)} = m_{\tau(2)} > m_{\tau(3)} > m_{\tau(4)}\}, \\ \{\mathbf{m} \mid m_{\tau(1)} > m_{\tau(2)} = m_{\tau(3)} > m_{\tau(4)}\}, & \quad \{\mathbf{m} \mid m_{\tau(1)} > m_{\tau(2)} > m_{\tau(3)} = m_{\tau(4)}\}, \\ \{\mathbf{m} \mid m_{\rho(1)} = m_{\rho(2)} > m_{\rho(3)} = m_{\rho(4)}\}, & \quad \{\mathbf{m} \mid m_{\nu(1)} = m_{\nu(2)} = m_{\nu(3)} > m_{\nu(4)}\}, \\ \{\mathbf{m} \mid m_{\nu(1)} > m_{\nu(2)} = m_{\nu(3)} = m_{\nu(4)}\}, & \quad \{\mathbf{m} \mid m_1 = m_2 = m_3 = m_4\}, \end{aligned}$$

where  $\sigma \in S, \tau \in A, \rho \in C \cup \overline{C}_{(14)}$  and  $\nu \in C$ . Similarly to (2.4) and (2.6), (2.10) follows from the decomposition of  $\sum_{m_1, m_2, m_3, m_4 > 0}$  induced by the one above.  $\square$

We prepare some equations in order to prove Proposition 2.1, which are obtained by summing up each harmonic relation in Lemma 2.2 with  $(l_1, l_2, l_3, l_4) = \sigma \cdot (l_1, l_2, l_3, l_4)$  for all  $\sigma \in C$ .

**LEMMA 2.3.** *Let  $l_1, l_2, l_3, l_4$  be positive integers.*

(i) *We have*

$$\begin{aligned} \sum_{\sigma \in C} \sigma \cdot [Li_{l_1, l_2, l_3}(\langle z \rangle^3) Li_{l_4}(z)] &= \left( 2 \sum_{\sigma \in C} + \sum_{\sigma \in C_{(12)}} + \sum_{\sigma \in C_{(34)}} \right) \sigma \cdot Li_{l_1, l_2, l_3, l_4}(\langle z \rangle^4) \\ &+ \left( \sum_{\sigma \in A} - \sum_{\sigma \in C_{(13)}} - \sum_{\sigma \in C_{(23)}} \right) \sigma \cdot [Li_{l_{12}, l_3, l_4}(z^2, z^3, z^4) + Li_{l_1, l_{23}, l_4}(z, z^3, z^4) \\ &\quad + Li_{l_1, l_2, l_{34}}(z, z^2, z^4)]. \quad (2.18) \end{aligned}$$

(ii) *We have*

$$\begin{aligned} \sum_{\sigma \in \overline{C}} \sigma \cdot [Li_{l_1, l_2}(\langle z \rangle^2) Li_{l_3, l_4}(\langle z \rangle^2)] &= \left( \sum_{\sigma \in C} + \sum_{\sigma \in C_{(14)}} + \sum_{\sigma \in C_{(23)}} \right) \sigma \cdot Li_{l_1, l_2, l_3, l_4}(\langle z \rangle^4) \\ &+ \sum_{\sigma \in C_{(23)}} \sigma \cdot [Li_{l_{12}, l_3, l_4}(z^2, z^3, z^4) + Li_{l_1, l_{23}, l_4}(z, z^3, z^4) + Li_{l_1, l_2, l_{34}}(z, z^2, z^4)] \\ &\quad + \sum_{\sigma \in \overline{C}_{(14)}} \sigma \cdot Li_{l_{12}, l_{34}}(z^2, z^4). \quad (2.19) \end{aligned}$$

(iii) *We have*

$$\sum_{\sigma \in C} \sigma \cdot [Li_{l_1, l_2}(\langle z \rangle^2) Li_{l_3}(z) Li_{l_4}(z)] = \left( 2 \sum_{\sigma \in S} + \sum_{\sigma \in C} - \sum_{\sigma \in C_{(13)}} \right) \sigma \cdot Li_{l_1, l_2, l_3, l_4}(\langle z \rangle^4)$$

$$\begin{aligned}
& + \left( 2 \sum_{\sigma \in A} - \sum_{\sigma \in C_{(13)}} \right) \sigma \cdot [Li_{l_1, l_2, l_3, l_4}(z^2, z^3, z^4) + Li_{l_1, l_2, l_3, l_4}(z, z^3, z^4) + Li_{l_1, l_2, l_3, l_4}(z, z^2, z^4)] \\
& + \left( \sum_{\sigma \in C} + 2 \sum_{\sigma \in \bar{C}_{(14)}} \right) \sigma \cdot Li_{l_1, l_2, l_3, l_4}(z^2, z^4) + \sum_{\sigma \in C} \sigma \cdot [Li_{l_1, l_2, l_3, l_4}(z^3, z^4) + Li_{l_1, l_2, l_3, l_4}(z, z^4)]. \quad (2.20)
\end{aligned}$$

*Proof.* We see from direct calculations that  $S$  is decomposed into the six right cosets

$$\begin{aligned}
C &= \{e, (1234), (13)(24), (1432)\}, & C_{(12)} &= \{(12), (134), (1423), (243)\}, \\
C_{(13)} &= \{(13), (14)(23), (24), (12)(34)\}, & C_{(14)} &= \{(14), (234), (1243), (132)\}, \quad (2.21) \\
C_{(23)} &= \{(23), (124), (1342), (143)\}, & C_{(34)} &= \{(34), (123), (1324), (142)\},
\end{aligned}$$

and from Table 1 that

$$\begin{aligned}
C &\equiv C_{(12)}, & C_{(13)} &\equiv C_{(34)}, & C_{(14)} &\equiv C_{(23)} & \text{in } S/\langle(12)\rangle, \\
C &\equiv C_{(23)}, & C_{(12)} &\equiv C_{(34)}, & C_{(13)} &\equiv C_{(14)} & \text{in } S/\langle(23)\rangle, \\
C &\equiv C_{(34)}, & C_{(12)} &\equiv C_{(13)}, & C_{(14)} &\equiv C_{(23)} & \text{in } S/\langle(34)\rangle. \quad (2.22)
\end{aligned}$$

By (2.21) we obtain

$$\sum_{\sigma \in C \cdot \mathcal{U}_1} = 2 \sum_{\sigma \in C} + \sum_{\sigma \in C_{(12)}} + \sum_{\sigma \in C_{(34)}}, \quad \sum_{\sigma \in C_{(243)}} = \sum_{\sigma \in C_{(12)}}.$$

Summing up (2.4) with  $(l_1, l_2, l_3, l_4) = \sigma \cdot (l_1, l_2, l_3, l_4)$  for  $\sigma \in C$  thus gives (2.18) since  $A$  is a transversal of  $S/\langle(12)\rangle$ ,  $S/\langle(23)\rangle$  or  $S/\langle(34)\rangle$ .

Similarly we find from direct calculations that  $\bar{C} \cdot \mathcal{V}_2 = C_{(23)}$  and  $\bar{C} \cdot \mathcal{U}_2 = C \cup C_{(14)} \cup C_{(23)}$ , and from Table 1 that  $\bar{C}_{(23)} \equiv \bar{C}_{(14)}$  in  $S/\langle(12), (34)\rangle$ . Therefore summing up (2.6) with  $(l_1, l_2, l_3, l_4) = \sigma \cdot (l_1, l_2, l_3, l_4)$  for  $\sigma \in \bar{C}$  gives (2.19).

We see from (2.21) that

$$\begin{aligned}
\sum_{\sigma \in C \cdot \mathcal{W}_3} &= \sum_{\sigma \in C_{(13)}} + \sum_{\sigma \in C_{(23)}}, \\
\sum_{\sigma \in C \cdot \mathcal{V}_3} &= \sum_{\sigma \in C} + \sum_{\sigma \in C_{(12)}} + \sum_{\sigma \in C_{(13)}} + 2 \sum_{\sigma \in C_{(23)}} + \sum_{\sigma \in C_{(34)}}, \\
\sum_{\sigma \in C \cdot \mathcal{U}_3} &= 2 \sum_{\sigma \in C} + \sum_{\sigma \in C} - \sum_{\sigma \in C_{(13)}},
\end{aligned}$$

and from Table 1 that  $C_{(13)} \equiv C$ ,  $C_{(23)} \equiv \bar{C}_{(14)}$  in  $S/\langle(12), (34)\rangle$  and  $C_{(24)} \equiv C$  in  $S/S^4$ . Because of (2.22), summing up (2.8) with  $(l_1, l_2, l_3, l_4) = \sigma \cdot (l_1, l_2, l_3, l_4)$  for  $\sigma \in C$  also gives (2.20).  $\square$

We prove Proposition 2.1.

*Proof of Proposition 2.1.* It follows from (2.10), (2.18), (2.19) and (2.20) that

$$\begin{aligned}
& \sum_{\sigma \in C} \sigma \cdot [Li_{l_1, l_2, l_3}(\langle z \rangle^3) Li_{l_4}(z) - Li_{l_1, l_2}(\langle z \rangle^2) Li_{l_3}(z) Li_{l_4}(z)] \\
& + \sum_{\sigma \in \bar{C}} \sigma \cdot [Li_{l_1, l_2}(\langle z \rangle^2) Li_{l_3, l_4}(\langle z \rangle^2) + Li_{l_1}(z) Li_{l_2}(z) Li_{l_3}(z) Li_{l_4}(z)] \quad (2.23)
\end{aligned}$$

$$= \sum_{\sigma \in C} \sigma \cdot Li_{l_1, l_2, l_3, l_4}(\langle z \rangle^4) + Li_{l_{1234}}(z^4).$$

Adding together (2.23) up to  $x_1^{l_1-1}x_2^{l_2-1}x_3^{l_3-1}x_4^{l_4-1}$  for all positive integers  $l_1, l_2, l_3, l_4$  with  $l_1 + l_2 + l_3 + l_4 = l$  proves (2.3).  $\square$

*REMARK 2.4.* Let  $\mathbf{l} = (l_1, l_2, l_3, l_4)$  be an index set of integers with  $l_j \geq 2$ . By (2.23) with  $z = 1$ , we obtain an equation for a cyclic sum of  $\zeta(\mathbf{l})$ ,

$$\begin{aligned} \sum_{\sigma \in C} \sigma \cdot \zeta(\mathbf{l}) &= \zeta(l_1)\zeta(l_2)\zeta(l_3)\zeta(l_4) + \sum_{\sigma \in \overline{C}} \sigma \cdot [\zeta(l_1, l_2)\zeta(l_3, l_4)] \\ &\quad + \sum_{\sigma \in C} \sigma \cdot [\zeta(l_1, l_2, l_3)\zeta(l_4) - \zeta(l_1, l_2)\zeta(l_3)\zeta(l_4)] - \zeta(l_{1234}). \end{aligned} \quad (2.24)$$

On the other hand, Hoffman [7, Theorem 2.2] proved equations for symmetric sums of multiple zeta values. The equations in cases of double, triple and quadruple zeta values are as follows. (Note that  $A$  instead of  $\zeta$  is used for a symbol of multiple zeta value in [7].)

$$\sum_{\sigma \in S_2} \sigma \cdot \zeta(\mathbf{l}_2) = \zeta(l_1)\zeta(l_2) - \zeta(l_{12}), \quad (2.25)$$

$$\sum_{\sigma \in S_3} \sigma \cdot \zeta(\mathbf{l}_3) = \zeta(l_1)\zeta(l_2)\zeta(l_3) - \sum_{\sigma \in \langle(123)\rangle} \sigma \cdot [\zeta(l_{12})\zeta(l_3)] + 2\zeta(l_{123}), \quad (2.26)$$

$$\begin{aligned} \sum_{\sigma \in S} \sigma \cdot \zeta(\mathbf{l}) &= \zeta(l_1)\zeta(l_2)\zeta(l_3)\zeta(l_4) - \sum_{\sigma \in C \cup \overline{C}_{(14)}} \sigma \cdot [\zeta(l_{12})\zeta(l_3)\zeta(l_4)] \\ &\quad + \sum_{\sigma \in \langle(123)\rangle} \sigma \cdot [\zeta(l_{12})\zeta(l_{34})] + 2 \sum_{\sigma \in C} \sigma \cdot [\zeta(l_{123})\zeta(l_4)] - 6\zeta(l_{1234}), \end{aligned} \quad (2.27)$$

where we put  $\mathbf{l}_2 = (l_1, l_2)$  and  $\mathbf{l}_3 = (l_1, l_2, l_3)$ .

By virtue of (2.25) and (2.26), summing up (2.24) with the left actions of  $\sigma \in \{e, (12), (13), (14), (23), (34)\}$  yields (2.27) after some calculations.

## 2.2 Identities derived from shuffle relations

The purpose of this subsection is to prove the following identities.

**PROPOSITION 2.5.** *Let  $\{z\}^n$  denote the  $n$ -tuple  $(z, \dots, z)$ , and  $l$  be an integer with  $l \geq 4$ .*

(i) *We have*

$$\begin{aligned} \sum_{\substack{a \geq 3, b \geq 1 \\ (a+b=l)}} \mathfrak{T}\mathfrak{L}_a(x_1, x_2, x_3; \{z\}^3) \mathfrak{S}\mathfrak{L}_b(x_4; z) &= \sum_{\rho \in \langle(34)\rangle} \mathfrak{Q}\mathfrak{L}_l(x_{14}, x_{24}, x_{\rho(3)\rho(4)}, x_{\rho(4)}; \{z\}^4) \\ &\quad + \mathfrak{Q}\mathfrak{L}_l(x_{14}, x_{42}, x_2, x_3; \{z\}^4) + \mathfrak{Q}\mathfrak{L}_l(x_{41}, x_1, x_2, x_3; \{z\}^4). \end{aligned} \quad (2.28)$$

(ii) *We have*

$$\sum_{\substack{a, b \geq 2 \\ (a+b=l)}} \mathfrak{D}\mathfrak{L}_a(x_1, x_2; \{z\}^2) \mathfrak{D}\mathfrak{L}_b(x_3, x_4; \{z\}^2) \quad (2.29)$$

$$= \sum_{\sigma \in \langle (13)(24) \rangle} \sigma \cdot \left[ \sum_{\rho \in \langle (24) \rangle} \mathfrak{Q}\mathfrak{L}_l(x_{13}, x_{32}, x_{\rho(2)\rho(4)}, x_{\rho(4)}; \{z\}^4) + \mathfrak{Q}\mathfrak{L}_l(x_{13}, x_{23}, x_3, x_4; \{z\}^4) \right].$$

(iii) We have

$$\begin{aligned} & \sum_{\substack{a \geq 2, b, c \geq 1 \\ (a+b+c=l)}} \mathfrak{D}\mathfrak{L}_a(x_1, x_2; \{z\}^2) \mathfrak{S}\mathfrak{L}_b(x_3; z) \mathfrak{S}\mathfrak{L}_c(x_4; z) \\ &= \sum_{\sigma \in \langle (34) \rangle} \sigma \cdot \left[ \sum_{\rho \in \langle (234) \rangle} \mathfrak{Q}\mathfrak{L}_l(x_{134}, x_{\rho(2)\rho(3)\rho(4)}, x_{\rho(3)\rho(4)}, x_{\rho(4)}; \{z\}^4) \right. \\ & \quad \left. + \sum_{\rho \in \langle (24) \rangle} \mathfrak{Q}\mathfrak{L}_l(x_{314}, x_{14}, x_{\rho(2)\rho(4)}, x_{\rho(4)}; \{z\}^4) + \mathfrak{Q}\mathfrak{L}_l(x_{341}, x_{41}, x_1, x_2; \{z\}^4) \right]. \end{aligned} \quad (2.30)$$

(iv) We have

$$\begin{aligned} & \sum_{\substack{a, b, c, d \geq 1 \\ (a+b+c+d=l)}} \mathfrak{S}\mathfrak{L}_a(x_1; z) \mathfrak{S}\mathfrak{L}_b(x_2; z) \mathfrak{S}\mathfrak{L}_c(x_3; z) \mathfrak{S}\mathfrak{L}_d(x_4; z) \\ &= \sum_{\sigma \in S} \sigma \cdot \mathfrak{Q}\mathfrak{L}_l(x_{1234}, x_{234}, x_{34}, x_4; \{z\}^4). \end{aligned} \quad (2.31)$$

As we see in the proof below, each coefficient of  $x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1}$ 's of the above identities is a shuffle relation for quadruple polylogarithms because of the equivalence between partial fraction expansions and shuffle relations (see [6] and [12]).

*Proof of Proposition 2.5.* By the partial fraction procedure (see [21, Lemma 2.3] for example), we obtain

$$\frac{1}{m_{123}m_{23}m_3m_4} = \sum_{\sigma \in \mathcal{U}_1} \frac{1}{m_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)}m_{\sigma(2)\sigma(3)\sigma(4)}m_{\sigma(3)\sigma(4)}m_{\sigma(4)}}$$

where  $\mathcal{U}_1$  is the set defined in (2.5). By replacing  $m_j$  by  $m_j - ty_j$  and calculating the coefficient of  $t^{l-4}$ , we get

$$\begin{aligned} & \sum_{\substack{a \geq 3, b \geq 1 \\ (a+b=l)}} \left( \sum_{\substack{l_1, l_2, l_3 \geq 1 \\ (l_1+l_2+l_3=a)}} \frac{y_{123}^{l_1-1} y_{23}^{l_2-1} y_3^{l_3-1}}{m_{123}^{l_1} m_{23}^{l_2} m_3^{l_3}} \right) \frac{y_4^{b-1}}{m_4^b} \\ &= \sum_{\sigma \in \mathcal{U}_1} \sum_{\substack{l_1, l_2, l_3, l_4 \geq 1 \\ (l_1+l_2+l_3+l_4=l)}} \frac{y_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)}^{l_1-1} y_{\sigma(2)\sigma(3)\sigma(4)}^{l_2-1} y_{\sigma(3)\sigma(4)}^{l_3-1} y_{\sigma(4)}^{l_4-1}}{m_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)}^{l_1} m_{\sigma(2)\sigma(3)\sigma(4)}^{l_2} m_{\sigma(3)\sigma(4)}^{l_3} m_{\sigma(4)}^{l_4}}. \quad (2.32) \end{aligned}$$

By (2.1) and (2.2), the sum of (2.32) up to  $z^{m_{1234}}$  for all positive integers  $m_1, m_2, m_3, m_4$  gives

$$\sum_{\substack{a \geq 3, b \geq 1 \\ (a+b=l)}} \mathfrak{T}\mathfrak{L}_a(y_{123}, y_{23}, y_3; \{z\}^3) \mathfrak{S}\mathfrak{L}_b(y_4; z) = \sum_{\sigma \in \mathcal{U}_1} \sigma \cdot \mathfrak{Q}\mathfrak{L}_l(y_{1234}, y_{234}, y_{34}, y_4; \{z\}^4),$$

which with  $(y_1, y_2, y_3, y_4) = (x_1 - x_2, x_2 - x_3, x_3, x_4)$  proves (2.28).

In the same way, we can find from the partial fraction expansion

$$\frac{1}{m_{12}m_2m_{34}m_4} = \sum_{\sigma \in \mathcal{U}_2} \frac{1}{m_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)}m_{\sigma(2)\sigma(3)\sigma(4)}m_{\sigma(3)\sigma(4)}m_{\sigma(4)}}$$

that

$$\sum_{\substack{a,b \geq 2 \\ (a+b=l)}} \mathfrak{DL}_a(y_{12}, y_2; \{z\}^2) \mathfrak{DL}_b(y_{34}, y_4; \{z\}^2) = \sum_{\sigma \in \mathcal{U}_2} \sigma \cdot \mathfrak{Q}\mathfrak{L}_l(y_{1234}, y_{234}, y_{34}, y_4; \{z\}^4),$$

which with  $(y_1, y_2, y_3, y_4) = (x_1 - x_2, x_2, x_3 - x_4, x_4)$  proves (2.29).

Identity (2.30) can be reduced to (2.29) by using

$$\sum_{\substack{b,c \geq 1 \\ (b+c=k)}} \mathfrak{SL}_b(x_3; z) \mathfrak{SL}_c(x_4; z) = \sum_{\sigma \in \langle (34) \rangle} \sigma \cdot \mathfrak{DL}_k(x_{34}, x_4; \{z\}^2)$$

for integers  $k \geq 2$ , which are obtained by the typical partial fraction expansion  $1/m_3m_4 = 1/m_{34}m_4 + 1/m_{43}m_3$ .

Identity (2.31) is proved by the partial fraction expansion

$$\frac{1}{m_1m_2m_3m_4} = \sum_{\sigma \in S} \frac{1}{m_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)}m_{\sigma(2)\sigma(3)\sigma(4)}m_{\sigma(3)\sigma(4)}m_{\sigma(4)}}$$

similarly to (2.28) and (2.29).  $\square$

## 2.3 Asymptotic properties

Let  $O$  denote the Landau symbol. For any function  $F(z)$  which has a polynomial  $P(T)$  and a positive number  $J > 0$ , and satisfies the asymptotic property

$$F(z) = P(-\log(1-z)) + O\left((1-z)(\log(1-z))^J\right) \quad (z \nearrow 1), \quad (2.33)$$

we define the constant term of  $P(T)$  by  $C_0(F(z))$ , that is,  $C_0(F(z)) = P(0)$ . Here  $z \nearrow 1$  means bringing  $z$  close to 1 under the condition  $0 < z < 1$ . Function  $C_0$  is well defined since  $P(T)$  is uniquely determined by (2.33).

In this subsection, we evaluate images of some functions written in terms of multiple polylogarithms under  $C_0$ , which enable us to calculate the constant terms of the asymptotic expansions of the functions appearing in Propositions 2.1 and 2.5.

Firstly we introduce the basic asymptotic properties about the multiple polylogarithms  $Li_{l_1, \dots, l_n}(\{z\}^n)$  which were shown in [11, §2].

**LEMMA 2.6** ([11]). *For any multiple polylogarithm  $Li_{l_1, \dots, l_n}(\{z\}^n)$ , there are a unique polynomial  $Z_{l_1, \dots, l_n}^{\text{m}}(T)$  and a positive number  $J > 0$  such that (2.33) holds.*

We call  $Z_{l_1, \dots, l_n}^{\text{m}}(0)$  the regularized multiple zeta value, and denote it by  $\zeta^{\text{m}}(l_1, \dots, l_n)$ . It is obvious by definition that

$$C_0(Li_{l_1, \dots, l_n}(\{z\}^n)) = \zeta^{\text{m}}(l_1, \dots, l_n). \quad (2.34)$$

For example,  $\zeta^{\text{m}}(l_1, \dots, l_n) = \zeta(l_1, \dots, l_n)$  for  $l_1 \geq 2$ , and  $\zeta^{\text{m}}(\{1\}^n) = 0$  because

$$Li_1(z) = -\log(1-z) \quad \text{and} \quad Li_{\{1\}^n}(\{z\}^n) = \frac{Li_1(z)^n}{n!}, \quad (2.35)$$

where the last equation is a shuffle relation derived from

$$\frac{1}{m_1 \cdots m_n} = \sum_{\sigma \in S_n} \frac{1}{m_{\sigma(1)} m_{\sigma(2)} \cdots m_{\sigma(n)} m_{\sigma(2)} \cdots m_{\sigma(n)} \cdots m_{\sigma(n)}}.$$

We define parameterized sums of regularized double, triple and quadruple zeta values by

$$\begin{aligned} \mathfrak{D}_l^{\text{m}}(x_1, x_2) &:= \sum^{\dagger} x_1^{l_1-1} x_2^{l_2-1} \zeta^{\text{m}}(\mathbf{l}), \\ \mathfrak{T}_l^{\text{m}}(x_1, x_2, x_3) &:= \sum^{\dagger} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} \zeta^{\text{m}}(\mathbf{l}), \\ \mathfrak{Q}_l^{\text{m}}(x_1, x_2, x_3, x_4) &:= \sum^{\dagger} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1} \zeta^{\text{m}}(\mathbf{l}), \end{aligned} \quad (2.36)$$

where  $\mathbf{l}$  means  $(l_1, \dots, l_n)$  for suitable  $n$ . We also denote  $x^{l-1} \zeta^{\text{m}}(l)$  by  $\mathfrak{S}_l^{\text{m}}(x)$  for convenience, and easily see that  $C_0(\mathfrak{Q}_l(x; z)) = \mathfrak{S}_l^{\text{m}}(x)$  by definition.

The objective images under  $C_0$  are in Propositions 2.7 and 2.8 below. The images in Proposition 2.7 (resp. 2.8) enable us to calculate the constant terms of the asymptotic expansions of the functions appearing in Proposition 2.1 (resp. 2.5).

**PROPOSITION 2.7.** *Let  $l$  be a positive integer. We assume that  $l \geq 2$  in the first,  $l \geq 3$  in the second, and  $l \geq 5$  in the third equation. Then we have*

$$C_0(\mathfrak{D}\mathfrak{L}_l(x_1, x_2; \langle z \rangle^2)) = \mathfrak{D}_l^{\text{m}}(x_1, x_2) + \begin{cases} 0 & (l > 2), \\ -\frac{\zeta(2)}{2} & (l = 2), \end{cases} \quad (2.37)$$

$$\begin{aligned} C_0(\mathfrak{T}\mathfrak{L}_l(x_1, x_2, x_3; \langle z \rangle^3)) &= \mathfrak{T}_l^{\text{m}}(x_1, x_2, x_3) \\ &+ \begin{cases} -\frac{\zeta(2)\zeta(l-2)}{2} x_3^{l-3} & (l > 3), \\ \frac{\zeta(3)}{3} & (l = 3), \end{cases} \end{aligned} \quad (2.38)$$

$$\begin{aligned} C_0(\mathfrak{Q}\mathfrak{L}_l(x_1, x_2, x_3, x_4; \langle z \rangle^4)) &= \mathfrak{Q}_l^{\text{m}}(x_1, x_2, x_3, x_4) \\ &- \frac{\zeta(2)}{2} \mathfrak{D}_{l-2}^{\text{m}}(x_3, x_4) + \frac{\zeta(3)\zeta(l-3)}{3} x_4^{l-4}. \end{aligned} \quad (2.39)$$

**PROPOSITION 2.8.** *Let  $l$  be an integer as in Proposition 2.7. Then we have*

$$C_0(\mathfrak{D}\mathfrak{L}_l(x_1, x_2; \{z\}^2)) = \mathfrak{D}_l^{\text{m}}(x_1, x_2), \quad (2.40)$$

$$C_0(\mathfrak{T}\mathfrak{L}_l(x_1, x_2, x_3; \{z\}^3)) = \mathfrak{T}_l^{\text{m}}(x_1, x_2, x_3), \quad (2.41)$$

$$C_0(\mathfrak{Q}\mathfrak{L}_l(x_1, x_2, x_3, x_4; \{z\}^4)) = \mathfrak{Q}_l^{\text{m}}(x_1, x_2, x_3, x_4). \quad (2.42)$$

By comparing Propositions 2.7 with 2.8, we see that the images of

$$\mathfrak{D}\mathfrak{L}_l(x_1, x_2; \langle z \rangle^2), \mathfrak{T}\mathfrak{L}_l(x_1, x_2, x_3; \langle z \rangle^3) \text{ and } \mathfrak{Q}\mathfrak{L}_l(x_1, x_2, x_3, x_4; \langle z \rangle^4)$$

under  $C_0$  are nearly equal to those of

$$\mathfrak{D}\mathfrak{L}_l(x_1, x_2; \{z\}^2), \mathfrak{T}\mathfrak{L}_l(x_1, x_2, x_3; \{z\}^3) \text{ and } \mathfrak{Q}\mathfrak{L}_l(x_1, x_2, x_3, x_4; \{z\}^4),$$

respectively, but they have some extra values.

We will prove the two propositions. Proposition 2.8 easily follows from (2.2), (2.34) and (2.36). We give two lemmas in order to prove Proposition 2.7.

**LEMMA 2.9.** *Let  $(l_1, \dots, l_n), (i_1, \dots, i_n)$  and  $(j_1, \dots, j_n)$  be  $n$ -tuples of positive integers. Assume that  $i_1 \leq \dots \leq i_n$ ,  $j_1 \leq \dots \leq j_n$  and  $i_a \leq j_a$  for every integer  $a$ . If there is a positive integer  $d$  such that  $d < n$ ,  $l_1 = \dots = l_d = 1$ ,  $l_{d+1} > 1$  and  $i_a = j_a$  ( $a = 1, \dots, d$ ), then*

$$\lim_{z \nearrow 1} (Li_{l_1, \dots, l_n}(z^{i_1}, \dots, z^{i_n}) - Li_{l_1, \dots, l_n}(z^{j_1}, \dots, z^{j_n})) = 0. \quad (2.43)$$

*Proof.* If  $i_a = j_a$  for any integer  $a$  with  $d+1 \leq a \leq n$ , then  $Li_{l_1, \dots, l_n}(z^{i_1}, \dots, z^{i_n}) = Li_{l_1, \dots, l_n}(z^{j_1}, \dots, z^{j_n})$  and (2.43) holds evidently. Suppose that it is false and  $b$  is the smallest integer such that  $i_b < j_b$ . We set  $i_0 = j_0 = 0$ , and put  $i'_a = i_a - i_{a-1}$  and  $j'_a = j_a - j_{a-1}$  for every integer  $a$ . From the conditions  $0 < i_1 \leq \dots \leq i_n$  and  $0 < j_1 \leq \dots \leq j_n$ , we easily see that  $i'_1, j'_1 > 0$  and  $i'_a, j'_a \geq 0$  for any integer  $a$  with  $2 \leq a \leq n$ . Since  $i'_a = j'_a$  ( $a = 1, \dots, b-1$ ) and  $i'_b < j'_b$  by the minimality of  $b$ , we have

$$\begin{aligned} & Li_{l_1, \dots, l_n}(z^{i_1}, \dots, z^{i_n}) - Li_{l_1, \dots, l_n}(z^{j_1}, \dots, z^{j_n}) \\ &= \sum_{m_1 > \dots > m_n > 0} \frac{z^{i'_1 m_1 + \dots + i'_n m_n} - z^{j'_1 m_1 + \dots + j'_n m_n}}{m_1^{l_1} \cdots m_n^{l_n}} \\ &= \sum_{m_1 > \dots > m_n > 0} \frac{z^{i'_1 m_1 + \dots + i'_b m_b} (z^{i'_{b+1} m_{b+1} + \dots + i'_n m_n} - z^{(j'_b - i'_b) m_b + j'_{b+1} m_{b+1} + \dots + j'_n m_n})}{m_1^{l_1} \cdots m_n^{l_n}}. \end{aligned}$$

We put  $i = i'_1$  and  $j = \max \{j'_b - i'_b, j'_{b+1}, \dots, j'_n\}$ . It is clear that  $i, j > 0$ . Under the condition  $0 < z < 1$ , we thus see that

$$\begin{aligned} 0 &< Li_{l_1, \dots, l_n}(z^{i_1}, \dots, z^{i_n}) - Li_{l_1, \dots, l_n}(z^{j_1}, \dots, z^{j_n}) \quad (2.44) \\ &\leq \sum_{m_1 > \dots > m_n > 0} \frac{z^{im_1} (1 - z^{(j'_b - i'_b) m_b + j'_{b+1} m_{b+1} + \dots + j'_n m_n})}{m_1^{l_1} \cdots m_n^{l_n}} \\ &\leq \sum_{m_1 > \dots > m_n > 0} \frac{z^{im_1} (1 - z^{(j'_b - i'_b + j'_{b+1} + \dots + j'_n) m_{d+1}})}{m_1^{l_1} \cdots m_n^{l_n}} \\ &< \sum_{m_1 > \dots > m_n > 0} \frac{z^{im_1} (1 - z^{nj m_{d+1}})}{m_1^{l_1} \cdots m_n^{l_n}}. \end{aligned}$$

Since  $1 - z^k = (1 - z) \sum_{h=0}^{k-1} z^h < k(1 - z)$  and  $l_{d+1} > 1$ , it follows that

$$\sum_{m_1 > \dots > m_n > 0} \frac{z^{im_1} (1 - z^{nj m_{d+1}})}{m_1^{l_1} \cdots m_n^{l_n}} \quad (2.45)$$

$$\begin{aligned}
&< nj(1-z) \sum_{m_1 > \dots > m_n > 0} \frac{z^{im_1}}{m_1^{l_1} \cdots m_{d+1}^{l_{d+1}-1} \cdots m_n^{l_n}} \\
&\leq nj(1-z) \sum_{m_1 > \dots > m_n > 0} \frac{z^{im_1}}{m_1 \cdots m_n} \\
&= nj(1-z) Li_{\{1\}^n}(\{z^i\}^n).
\end{aligned}$$

We see from (2.35) that

$$\lim_{z \nearrow 1} (1-z) Li_{\{1\}^n}(\{z^i\}^n) = 0,$$

which with (2.44) and (2.45) proves (2.43).  $\square$

We note the following facts; When  $l_1, \dots, l_n$  are positive integers,  $l_1 \cdots l_n = 1$  if and only if  $l_1 = \dots = l_n = 1$ , and  $l_1 \cdots l_n > 1$  if and only if  $l_j > 1$  for some  $j$ .

**LEMMA 2.10.** *Let  $l_1, l_2, l_3, l_4, l$  be positive integers. We assume that  $l = l_1 + l_2 + l_3$  in the second, and  $l = l_1 + l_2 + l_3 + l_4$  and  $l \geq 5$  in the third equation. Then we have*

$$C_0(Li_{l_1, l_2}(\langle z \rangle^2)) = \zeta^{\text{m}}(l_1, l_2) + \begin{cases} 0 & (l_1 l_2 > 1), \\ -\frac{\zeta(2)}{2} & (l_1 l_2 = 1), \end{cases} \quad (2.46)$$

$$\begin{aligned}
C_0(Li_{l_1, l_2, l_3}(\langle z \rangle^3)) &= \zeta^{\text{m}}(l_1, l_2, l_3) \\
&+ \begin{cases} 0 & (l_1 l_2 > 1), \\ -\frac{\zeta(2)\zeta(l-2)}{2} & (l_1 l_2 = 1 \text{ and } l_3 > 1), \\ \frac{\zeta(3)}{3} & (l_1 l_2 l_3 = 1), \end{cases} \quad (2.47)
\end{aligned}$$

$$\begin{aligned}
C_0(Li_{l_1, l_2, l_3, l_4}(\langle z \rangle^4)) &= \zeta^{\text{m}}(l_1, l_2, l_3, l_4) \\
&+ \begin{cases} 0 & (l_1 l_2 > 1), \\ -\frac{\zeta(2)\zeta(l_3, l_4)}{2} & (l_1 l_2 = 1 \text{ and } l_3 > 1), \\ \frac{\zeta(3)\zeta(l-3)}{3} - \frac{\zeta(2)\zeta^{\text{m}}(1, l-3)}{2} & (l_1 l_2 l_3 = 1 \text{ and } l_4 > 1). \end{cases} \quad (2.48)
\end{aligned}$$

*Proof.* Strictly speaking, it is necessary to verify the fact that the polylogarithms in the lemma have asymptotic properties such as (2.33). This fact is easily seen in the courses of the proofs below by virtue of Lemmas 2.6 and 2.9, and we do not mention it anymore.

Equations (2.46) and (2.47) except the case of  $l_1 l_2 l_3 = 1$  are proved in [13, Lemma 2.4], and we omit their proofs. The remaining case is derived as follows. We see from (2.43) that  $C_0(Li_{1,2}(z, z^3)) = C_0(Li_{1,2}(z, z)) = \zeta^{\text{m}}(1, 2)$ . By using the harmonic relation

$$Li_1(z) Li_1(z) Li_1(z) = 6Li_{1,1,1}(z, z^2, z^3) + 3Li_{2,1}(z^2, z^3) + 3Li_{1,2}(z, z^3) + Li_3(z^3)$$

derived from the decomposition of  $\sum_{m_1, m_2, m_3 > 0}$ , we thus obtain

$$6C_0(Li_{1,1,1}(z, z^2, z^3)) = -3\zeta(2, 1) - 3\zeta^{\text{m}}(1, 2) - \zeta(3) = -3\zeta^{\text{m}}(1, 2) - 4\zeta(3),$$

where we used the simplest sum formula  $\zeta(2, 1) = \zeta(3)$ . This proves (2.47) for  $l_1 l_2 l_3 = 1$  since  $\zeta^{\text{m}}(1, 2) = -2\zeta(3)$  which follows from the image of the harmonic relation

$$Li_1(z) Li_{k-1}(z) = Li_{1,k-1}(z, z^2) + Li_{k-1,1}(z, z^2) + Li_k(z^2) \quad (2.49)$$

under  $C_0$  for  $k = 3$ .

Equation (2.48) for  $l_1 l_2 > 1$  is clear because of (2.34) and (2.43).

We prove (2.48) for  $l_1 l_2 = 1$  and  $l_3 > 1$ . By (2.12) with  $z_1 = z_2 = z_3 = z_4 = z$ , we obtain

$$\begin{aligned} Li_{1,1}(z, z) Li_{l_3, l_4}(z, z) &= Li_{1,1,l_3, l_4}(z, z, z^2, z^2) + Li_{1,l_3,1, l_4}(z, z^2, z^2, z^2) \\ &+ Li_{1,l_3, l_4,1}(z, z^2, z^2, z^2) + Li_{l_3,1,1, l_4}(z, z^2, z^2, z^2) + Li_{l_3,1, l_4,1}(z, z^2, z^2, z^2) \\ &+ Li_{l_3, l_4,1,1}(z, z, z^2, z^2) + Li_{l_3+1,1, l_4}(z^2, z^2, z^2) + Li_{l_3+1, l_4,1}(z^2, z^2, z^2) \\ &+ Li_{1,l_3+1, l_4}(z, z^2, z^2) + Li_{l_3, l_4+1,1}(z, z^2, z^2) + Li_{1,l_3, l_4+1}(z, z^2, z^2) \\ &+ Li_{l_3,1, l_4+1}(z, z^2, z^2) + Li_{l_3+1, l_4+1}(z^2, z^2). \end{aligned}$$

From this, (2.34) and (2.43), it follows that

$$\begin{aligned} \zeta^{\text{m}}(1, 1, l_3, l_4) &= -(\zeta^{\text{m}}(1, l_3, 1, l_4) + \zeta^{\text{m}}(1, l_3, l_4, 1) + \zeta^{\text{m}}(l_3, 1, 1, l_4) + \zeta^{\text{m}}(l_3, 1, l_4, 1) \\ &+ \zeta^{\text{m}}(l_3, l_4, 1, 1) + \zeta^{\text{m}}(l_3 + 1, 1, l_4) + \zeta^{\text{m}}(l_3 + 1, l_4, 1) + \zeta^{\text{m}}(1, l_3 + 1, l_4) \\ &+ \zeta^{\text{m}}(l_3, l_4 + 1, 1) + \zeta^{\text{m}}(1, l_3, l_4 + 1) + \zeta^{\text{m}}(l_3, 1, l_4 + 1) + \zeta^{\text{m}}(l_3 + 1, l_4 + 1)). \end{aligned} \quad (2.50)$$

On the other hand, by (2.6), we obtain

$$\begin{aligned} Li_{1,1}(\langle z \rangle^2) Li_{l_3, l_4}(\langle z \rangle^2) &= Li_{1,1,l_3, l_4}(\langle z \rangle^4) + Li_{1,l_3,1, l_4}(\langle z \rangle^4) + Li_{1,l_3, l_4,1}(\langle z \rangle^4) \\ &+ Li_{l_3,1,1, l_4}(\langle z \rangle^4) + Li_{l_3,1, l_4,1}(\langle z \rangle^4) + Li_{l_3, l_4,1,1}(\langle z \rangle^4) + Li_{l_3+1,1, l_4}(z^2, z^3, z^4) \\ &+ Li_{l_3+1, l_4,1}(z^2, z^3, z^4) + Li_{1,l_3+1, l_4}(z, z^3, z^4) + Li_{l_3, l_4+1,1}(z, z^3, z^4) \\ &+ Li_{1,l_3, l_4+1}(z, z^2, z^4) + Li_{l_3,1, l_4+1}(z, z^2, z^4) + Li_{l_3+1, l_4+1}(z^2, z^4). \end{aligned}$$

From this, (2.34), (2.43), (2.46) and (2.50), it also follows that

$$C_0(Li_{1,1,l_3, l_4}(\langle z \rangle^4)) = \zeta^{\text{m}}(1, 1, l_3, l_4) - \frac{\zeta(2)\zeta(l_3, l_4)}{2}.$$

This proves (2.48) for  $l_1 l_2 = 1$  and  $l_3 > 1$ .

Noting  $\zeta^{\text{m}}(1, l-3) = -(\zeta(l-3, 1) + \zeta(l-2))$  which is obtained by (2.34), (2.43) and (2.49) with  $k = l-2$ , we can similarly prove (2.48) for  $l_1 l_2 l_3 = 1$  and  $l_4 > 1$  by the use of (2.11) with  $z_1 = z_2 = z_3 = z_4 = z$  and (2.4). We thus omit the proof.  $\square$

We prove Proposition 2.7.

*Proof of Proposition 2.7.* Equations (2.37) and (2.38) are derived from (2.46) and (2.47), respectively. It is seen from (2.48) that

$$\begin{aligned} &C_0(\mathfrak{Q}\mathfrak{L}_l(x_1, x_2, x_3, x_4; \langle z \rangle^4)) \\ &= \mathfrak{Q}_l^{\text{m}}(x_1, x_2, x_3, x_4) - \frac{\zeta(2)}{2} \sum_{\substack{l_3 \geq 2, l_4 \geq 1 \\ (l_3 + l_4 = l-2)}} \zeta(l_3, l_4) x_3^{l_3-1} x_4^{l_4-1} \end{aligned}$$

$$\begin{aligned}
& + \left( \frac{\zeta(3)\zeta(l-3)}{3} - \frac{\zeta(2)\zeta^{\text{m}}(1, l-3)}{2} \right) x_4^{l-4} \\
& = \mathfrak{Q}_l^{\text{m}}(x_1, x_2, x_3, x_4) - \frac{\zeta(2)}{2} \sum_{\substack{l_3, l_4 \geq 1 \\ (l_3 + l_4 = l-2)}} \zeta^{\text{m}}(l_3, l_4) x_3^{l_3-1} x_4^{l_4-1} + \frac{\zeta(3)\zeta(l-3)}{3} x_4^{l-4},
\end{aligned}$$

which proves (2.39).  $\square$

### 3 Proof of Theorem 1.1

We give a proof of Theorem 1.1 by using Propositions 2.1, 2.5, 2.7 and 2.8 above, and Lemmas 3.1, 3.2 and 3.3 below. We will show the lemmas after the proof.

*Proof of Theorem 1.1.* Propositions 2.1 and 2.7 yield

$$\begin{aligned}
& - \sum_{\sigma \in C} \sigma \cdot \mathfrak{Q}_l^{\text{m}}(x_1, x_2, x_3, x_4) + \sum_{\substack{a \geq 3, b \geq 1 \\ (a+b=l)}} \sum_{\sigma \in C} \sigma \cdot [\mathfrak{T}_a^{\text{m}}(x_1, x_2, x_3) \mathfrak{S}_b^{\text{m}}(x_4)] \\
& + \sum_{\substack{a, b \geq 2 \\ (a+b=l)}} \sum_{\sigma \in \overline{C}} \sigma \cdot [\mathfrak{D}_a^{\text{m}}(x_1, x_2) \mathfrak{D}_b^{\text{m}}(x_3, x_4)] - \sum_{\substack{a \geq 2, b, c \geq 1 \\ (a+b+c=l)}} \sum_{\sigma \in C} \sigma \cdot [\mathfrak{D}_a^{\text{m}}(x_1, x_2) \mathfrak{S}_b^{\text{m}}(x_3) \mathfrak{S}_c^{\text{m}}(x_4)] \\
& + \sum_{\substack{a, b, c, d \geq 1 \\ (a+b+c+d=l)}} \mathfrak{S}_a^{\text{m}}(x_1) \mathfrak{S}_b^{\text{m}}(x_2) \mathfrak{S}_c^{\text{m}}(x_3) \mathfrak{S}_d^{\text{m}}(x_4) \\
& = \left( \sum_{\substack{l_1, l_2, l_3, l_4 \geq 1 \\ (l_1 + l_2 + l_3 + l_4 = l)}} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1} \right) \zeta(l).
\end{aligned} \tag{3.1}$$

(The all extra values appearing in Proposition 2.7 are canceled each other.) We find from (3.1), Propositions 2.5 and 2.8 that

$$\begin{aligned}
& \sum_{\sigma \in S} \sigma \cdot \mathfrak{Q}_l^{\text{m}}(x_{1234}, x_{234}, x_{34}, x_4) \\
& - \sum_{\sigma \in C \cup C_{(34)}} \sigma \cdot \left[ \sum_{\rho \in \langle (234) \rangle} \mathfrak{Q}_l^{\text{m}}(x_{134}, x_{\rho(2)\rho(3)\rho(4)}, x_{\rho(3)\rho(4)}, x_{\rho(4)}) \right. \\
& \quad \left. + \sum_{\rho \in \langle (24) \rangle} \mathfrak{Q}_l^{\text{m}}(x_{314}, x_{14}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) + \mathfrak{Q}_l^{\text{m}}(x_{341}, x_{41}, x_1, x_2) \right] \\
& + \sum_{\sigma \in C} \sigma \cdot \left[ \sum_{\rho \in \langle (24) \rangle} \mathfrak{Q}_l^{\text{m}}(x_{13}, x_{32}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) + \sum_{\rho \in \langle (34) \rangle} \mathfrak{Q}_l^{\text{m}}(x_{14}, x_{24}, x_{\rho(3)\rho(4)}, x_{\rho(4)}) \right. \\
& \quad \left. + \mathfrak{Q}_l^{\text{m}}(x_{13}, x_{23}, x_3, x_4) + \mathfrak{Q}_l^{\text{m}}(x_{14}, x_{42}, x_2, x_3) + \mathfrak{Q}_l^{\text{m}}(x_{41}, x_1, x_2, x_3) \right. \\
& \quad \left. - \mathfrak{Q}_l^{\text{m}}(x_1, x_2, x_3, x_4) \right] \\
& = \left( \sum_{\substack{l_1, l_2, l_3, l_4 \geq 1 \\ (l_1 + l_2 + l_3 + l_4 = l)}} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1} \right) \zeta(l).
\end{aligned}$$

By Lemmas 3.1 and 3.2, this equation holds if we replace  $\mathfrak{Q}_l^{\text{III}}$  by  $\mathfrak{Q}_l$ , where  $\mathfrak{Q}_l(x_1, x_2, x_3, x_4)$  is a parameterized sum of multiple zeta values of weight  $l$  which is defined by

$$\mathfrak{Q}_l(x_1, x_2, x_3, x_4) := \sum' x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1} \zeta(l_1, l_2, l_3, l_4). \quad (3.2)$$

We thus see from Lemma 3.3 that

$$\begin{aligned} & \sum_{\sigma \in S} \sigma \cdot \mathfrak{Q}_l(x_{1234}, x_{234}, x_{34}, x_4) \\ & - \sum_{\sigma \in C \cup C_{(34)}} \sigma \cdot \left[ \sum_{\rho \in \langle (234) \rangle} \mathfrak{Q}_l(x_{134}, x_{\rho(2)\rho(3)\rho(4)}, x_{\rho(3)\rho(4)}, x_{\rho(4)}) \right. \\ & \quad \left. + \sum_{\rho \in \langle (24) \rangle} \mathfrak{Q}_l(x_{314}, x_{14}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) + \mathfrak{Q}_l(x_{341}, x_{41}, x_1, x_2) \right] \quad (3.3) \\ & + \sum_{\sigma \in C} \sigma \cdot \left[ \sum_{\rho \in \langle (\nu(2)4) \rangle} \sum_{\nu \in \bar{C}} \mathfrak{Q}_l(x_{1\nu(3)}, x_{\nu(3)2}, x_{\rho\nu(2)\rho(4)}, x_{\rho(4)}) \right. \\ & \quad \left. + \sum_{\nu \in \bar{C}} \mathfrak{Q}_l(x_{\nu(1)3}, x_{2\nu(3)}, x_{\nu(3)}, x_{\nu(4)}) + \mathfrak{Q}_l(x_{41}, x_1, x_2, x_3) - \mathfrak{Q}_l(x_1, x_2, x_3, x_4) \right] \\ & = \left( \sum_{\substack{l_1, l_2, l_3, l_4 \geq 1 \\ (l_1 + l_2 + l_3 + l_4 = l)}} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1} \right) \zeta(l), \end{aligned}$$

which with (3.2) proves (1.3).  $\square$

We show the lemmas.

**LEMMA 3.1.** *We have*

$$\mathfrak{Q}_l^{\text{III}}(x_1, x_2, x_3, x_4) = \mathfrak{Q}_l(x_1, x_2, x_3, x_4) + \mathfrak{Q}_l^{\text{III}}(0, x_2, x_3, x_4). \quad (3.4)$$

*Proof.* Equation (3.4) is obvious because of (2.36) and (3.2).  $\square$

**LEMMA 3.2.** *We have*

$$\begin{aligned} \sum_{\sigma \in S} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{234}, x_{34}, x_4) &= \sum_{\substack{\sigma \in C \cup C_{(34)} \\ \rho \in \langle (234) \rangle}} \sigma \rho \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{234}, x_{34}, x_4), \\ \sum_{\substack{\sigma \in C \cup C_{(34)} \\ \rho \in \langle (24) \rangle}} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{14}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) &= \sum_{\sigma \in C} \sigma \cdot \left[ \sum_{\rho \in \langle (24) \rangle} \mathfrak{Q}_l^{\text{III}}(0, x_{32}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) \right. \\ & \quad \left. + \sum_{\rho \in \langle (34) \rangle} \mathfrak{Q}_l^{\text{III}}(0, x_{24}, x_{\rho(3)\rho(4)}, x_{\rho(4)}) \right], \\ \sum_{\sigma \in C \cup C_{(34)}} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{41}, x_1, x_2) &= \sum_{\sigma \in C} \sigma \cdot \left[ \mathfrak{Q}_l^{\text{III}}(0, x_{23}, x_3, x_4) + \mathfrak{Q}_l^{\text{III}}(0, x_{42}, x_2, x_3) \right], \\ \sum_{\sigma \in C} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_1, x_2, x_3) &= \sum_{\sigma \in C} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_2, x_3, x_4). \end{aligned}$$

*Proof.* We use (2.21) without notice. We obtain the first equation by  $(C \cup C_{(34)}) \cdot \langle(234)\rangle = C \cup C_{(234)} \cup C_{(243)} \cup C_{(34)} \cup C_{(24)} \cup C_{(23)} = S$ . Since  $C = C_{(13)(24)}$  and  $C_{(34)} = C_{(123)}$ , we have

$$\begin{aligned} \sum_{\substack{\sigma \in C \\ \rho \in \langle(24)\rangle}} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{14}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) &= \sum_{\substack{\sigma \in C_{(13)(24)} \\ \rho \in \langle(24)\rangle}} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{14}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) \\ &= \sum_{\substack{\sigma \in C \\ \rho \in (13)(24) \cdot \langle(24)\rangle}} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{32}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) \\ &= \sum_{\substack{\sigma \in C \\ \rho \in \langle(24)\rangle}} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{32}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) \end{aligned}$$

and

$$\begin{aligned} \sum_{\substack{\sigma \in C_{(34)} \\ \rho \in \langle(24)\rangle}} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{14}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) &= \sum_{\substack{\sigma \in C_{(123)} \\ \rho \in \langle(24)\rangle}} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{14}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) \\ &= \sum_{\substack{\sigma \in C \\ \rho \in (123) \cdot \langle(24)\rangle}} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{24}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) \\ &= \sum_{\substack{\sigma \in C \\ \rho \in \langle(34)\rangle}} \sigma \cdot \mathfrak{Q}_l^{\text{III}}(0, x_{24}, x_{\rho(3)\rho(4)}, x_{\rho(4)}), \end{aligned}$$

which gives the second one. Similarly the third one follows. The fourth one is derived from  $C = C_{(1234)}$ .  $\square$

**LEMMA 3.3.** *We have*

$$\begin{aligned} \sum_{\sigma \in C} \sigma \cdot \left[ \sum_{\rho \in \langle(24)\rangle} \mathfrak{Q}_l(x_{13}, x_{32}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) + \sum_{\rho \in \langle(34)\rangle} \mathfrak{Q}_l(x_{14}, x_{24}, x_{\rho(3)\rho(4)}, x_{\rho(4)}) \right] \\ = \sum_{\sigma \in C} \sigma \cdot \left[ \sum_{\rho \in \langle(\nu(2)4)\rangle} \sum_{\nu \in \overline{C}} \mathfrak{Q}_l(x_{1\nu(3)}, x_{\nu(3)2}, x_{\rho\nu(2)\rho(4)}, x_{\rho(4)}) \right] \quad (3.5) \end{aligned}$$

and

$$\begin{aligned} \sum_{\sigma \in C} \sigma \cdot [\mathfrak{Q}_l(x_{13}, x_{23}, x_3, x_4) + \mathfrak{Q}_l(x_{14}, x_{42}, x_2, x_3)] \\ = \sum_{\sigma \in C} \sigma \cdot \left[ \sum_{\nu \in \overline{C}} \mathfrak{Q}_l(x_{\nu(1)3}, x_{2\nu(3)}, x_{\nu(3)}, x_{\nu(4)}) \right]. \quad (3.6) \end{aligned}$$

*Proof.* Since  $\overline{C} = \{e, (1234)\}$ , we have

$$\begin{aligned} \sum_{\rho \in \langle(24)\rangle} \mathfrak{Q}_l(x_{13}, x_{32}, x_{\rho(2)\rho(4)}, x_{\rho(4)}) + \sum_{\rho \in \langle(34)\rangle} \mathfrak{Q}_l(x_{14}, x_{24}, x_{\rho(3)\rho(4)}, x_{\rho(4)}) \\ = \sum_{\rho \in \langle(\nu(2)4)\rangle} \sum_{\nu \in \overline{C}} \mathfrak{Q}_l(x_{1\nu(3)}, x_{\nu(3)2}, x_{\rho\nu(2)\rho(4)}, x_{\rho(4)}), \end{aligned}$$

which gives (3.5). We also have

$$\mathfrak{Q}_l(x_{13}, x_{23}, x_3, x_4) + \mathfrak{Q}_l(x_{32}, x_{24}, x_4, x_1) = \sum_{\nu \in \overline{C}} \mathfrak{Q}_l(x_{\nu(1)3}, x_{2\nu(3)}, x_{\nu(3)}, x_{\nu(4)}).$$

This proves (3.6) because of

$$\sum_{\sigma \in C} \sigma \cdot \mathfrak{Q}_l(x_{32}, x_{24}, x_4, x_1) = \sum_{\sigma \in C} \sigma \cdot \mathfrak{Q}_l(x_{14}, x_{42}, x_2, x_3)$$

which follows from  $C = C_{(13)(24)}$ .  $\square$

## 4 Proof of Theorem 1.2

In this final section, we derive Theorem 1.2 from Theorem 1.1. Before proving Theorem 1.2, we prepare some equations by substituting 0 or 1 for each parameter  $x_j$  in (1.3). For the substitutions, the mathematical software ‘‘Maxima’’ is used implicitly, and (3.3) instead of (1.3) is referred to since both are equivalent and (3.3) is convenient to calculate.

**LEMMA 4.1.** *Let  $l$  be a positive integer with  $l \geq 5$ . We have*

$$\begin{aligned} 2\mathfrak{Q}_l(2, 2, 2, 1) + \mathfrak{Q}_l(2, 2, 1, 1) + \mathfrak{Q}_l(2, 1, 1, 1) \\ - 2\mathfrak{Q}_l(1, 2, 2, 1) - \mathfrak{Q}_l(1, 2, 1, 1) - 3\mathfrak{Q}_l(1, 1, 1, 1) \end{aligned} = (l-3)\zeta(l), \quad (4.1)$$

$$\begin{aligned} 4\mathfrak{Q}_l(2, 2, 2, 1) + 2\mathfrak{Q}_l(2, 2, 1, 1) - 4\mathfrak{Q}_l(1, 2, 2, 1) \\ - 2\mathfrak{Q}_l(1, 2, 1, 1) - 4\mathfrak{Q}_l(1, 1, 2, 1) \end{aligned} = (l-3)\zeta(l), \quad (4.2)$$

$$\begin{aligned} 6\mathfrak{Q}_l(3, 3, 2, 1) + 4\mathfrak{Q}_l(3, 2, 2, 1) + 2\mathfrak{Q}_l(3, 2, 1, 1) \\ - 6\mathfrak{Q}_l(2, 3, 2, 1) - 8\mathfrak{Q}_l(2, 2, 2, 1) - 4\mathfrak{Q}_l(2, 2, 1, 1) \\ - 2\mathfrak{Q}_l(2, 1, 2, 1) - 2\mathfrak{Q}_l(2, 1, 1, 1) + 4\mathfrak{Q}_l(1, 2, 2, 1) \\ + 2\mathfrak{Q}_l(1, 2, 1, 1) + 2\mathfrak{Q}_l(1, 1, 2, 1) + 3\mathfrak{Q}_l(1, 1, 1, 1) \end{aligned} = \binom{l-2}{2}\zeta(l), \quad (4.3)$$

$$\begin{aligned} 24\mathfrak{Q}_l(4, 3, 2, 1) - 24\mathfrak{Q}_l(3, 3, 2, 1) - 16\mathfrak{Q}_l(3, 2, 2, 1) \\ - 8\mathfrak{Q}_l(3, 2, 1, 1) + 16\mathfrak{Q}_l(2, 2, 2, 1) + 8\mathfrak{Q}_l(2, 2, 1, 1) \\ + 4\mathfrak{Q}_l(2, 1, 1, 1) - 4\mathfrak{Q}_l(1, 1, 1, 1) \end{aligned} = \binom{l-1}{3}\zeta(l). \quad (4.4)$$

*Proof.* Equations (4.1), (4.2), (4.3) and (4.4) are obtained by substituting  $(1, 1, 0, 0)$ ,  $(1, 0, 1, 0)$ ,  $(1, 1, 1, 0)$  and  $(1, 1, 1, 1)$  for  $(x_1, x_2, x_3, x_4)$  in (3.3), respectively.  $\square$

We prove Theorem 1.2. Note that  $\mathfrak{Q}_l(1, 1, 1, 1) = \zeta(l)$  by (1.2) with  $n = 4$ .

*Proof of Theorem 1.2.* Formulas (1.4) and (1.5) respectively follow from (4.1) and (4.2) because of (3.2). Formula (1.6) is derived from subtracting (1.5) from twice (1.4).

We prove (1.7) next. Since  $3(l-3)/2 + \binom{l-2}{2} = (l+1)(l-3)/2$ , adding up (4.1), the half of (4.2), and (4.3) yields

$$6\mathfrak{Q}_l(3, 3, 2, 1) + 4\mathfrak{Q}_l(3, 2, 2, 1) + 2\mathfrak{Q}_l(3, 2, 1, 1) - 6\mathfrak{Q}_l(2, 3, 2, 1) - 4\mathfrak{Q}_l(2, 2, 2, 1) \\ - 2\mathfrak{Q}_l(2, 2, 1, 1) - 2\mathfrak{Q}_l(2, 1, 2, 1) - \mathfrak{Q}_l(2, 1, 1, 1) = \frac{(l+1)(l-3)}{2}\zeta(l). \quad (4.5)$$

Since  $(l+1)(l-3)/2 + \binom{l-1}{3}/4 + 1 = (l+1)(l^2 + 5l - 18)/24$ , adding up (4.5) and the quarter of (4.4) also yields

$$6\mathfrak{Q}_l(4, 3, 2, 1) - 6\mathfrak{Q}_l(2, 3, 2, 1) - 2\mathfrak{Q}_l(2, 1, 2, 1) = \frac{(l+1)(l^2 + 5l - 18)}{24}\zeta(l),$$

which proves (1.7).  $\square$

*REMARK 4.2.* We rewrite (4.3), (4.4) and (4.5), which are necessary to prove (1.7), in terms of quadruple zeta values  $\zeta(\mathbf{l}) = \zeta(l_1, l_2, l_3, l_4)$  to see the explicit relations among these values.

$$\sum' (3^{l_{12}-1}2^{l_3} + 3^{l_1-1}2^{l_{23}} + 3^{l_1-1}2^{l_2} \\ - 3^{l_2}2^{l_{13}-1} - 2^{l_{123}} - 2^{l_{12}} - 2^{l_{13}-1} - 2^{l_1}) = \frac{l(l-5)}{2}\zeta(l), \quad (4.6)$$

$$+ 2^{l_{23}} + 2^{l_2} + 2^{l_3})\zeta(\mathbf{l})$$

$$\sum' (3^{l_2}2^{2l_1+l_3-1} - 3^{l_{12}-1}2^{l_3+1} - 3^{l_1-1}2^{l_{23}+1} \\ - 3^{l_1-1}2^{l_2+1} + 2^{l_{123}} + 2^{l_{12}} + 2^{l_1})\zeta(\mathbf{l}) = \frac{(l+1)(l^2 - 7l + 18)}{12}\zeta(l), \quad (4.7)$$

$$\sum' (3^{l_{12}-1}2^{l_3+1} + 3^{l_1-1}2^{l_{23}+1} + 3^{l_1-1}2^{l_2+1} \\ - 3^{l_2}2^{l_{13}} - 2^{l_{123}} - 2^{l_{12}} - 2^{l_{13}} - 2^{l_1})\zeta(\mathbf{l}) = (l+1)(l-3)\zeta(l), \quad (4.8)$$

where (4.6), (4.7) and (4.8) correspond to (4.3), (4.4) and (4.5), respectively.

## Acknowledgements

The author would like to thank the National Center for Theoretical Sciences (Taiwan) for the hospitality and support in his visit for “2012 NCTS Japan-Taiwan Joint Conference on Number Theory”. The author is also thankful to Professor Wen-Chin Liaw for introducing him the preprint [19].

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